Available at www.MATHEMATICSweb.org
POWERED BY SCIENCE @ DIRECT

LINEAR ALGEBRA AND ITS APPLICATIONS

ELSEVIER

Linear Algebra and its Applications 360 (2003) 59–81

www.elsevier.com/locate/laa

On supermatrix idempotent operator semigroups

Steven Duplij

Theory Group, Nuclear Physics Laboratory, Kharkov National University, Kharkov 61077, Ukraine

Received 6 June 2000; accepted 11 May 2002

Submitted by L. Rodman

Abstract

One-parameter semigroups of antitriangle idempotent supermatrices and corresponding superoperator semigroups are introduced and investigated. It is shown that t-linear idempotent superoperators and exponential superoperators are mutually dual in some sense, the first giving rise to additional non-exponential solutions to the initial Cauchy problem. The corresponding functional equation and analog of resolvent are found. Differential and functional equations for idempotent (super)operators are derived for their general t power-type dependence.

AMS classification: 47D06; 58A50; 81T60

Keywords: Semigroup; Supermatrix; Grassmann algebra; Idempotent; Nilpotence; Zero-divisor; Cauchy problem; Resolvent; Band

1. Introduction

The operator semigroups [1] play an important role in mathematical physics [2–4] providing a theoretical framework for evolving systems [5–7]. Although developments in this area have encompassed many new fields [8–11], until recently there have been no attempts to apply operator semigroups to the important topic of supersymmetry [12–14]. The main new feature in the mathematics of supersymmetry [15–17] is the existence of noninvertible objects, zero-divisors and nilpotents, as elements of matrices. Taking into account that such objects form a semigroup, we call our study that of the *semigroup*$\times$*semigroup* method.

E-mail address: steven.a.duplij@univer.kharkov.ua
URL: http://www.math.uni-mannheim.de/∼duplij

0024-3795/03/$ - see front matter
PII: S0024-3795(02)00444-5

Here we investigate continuous supermatrix representations of idempotent operator semigroups firstly introduced for bands in [18,19] and then studied in [20,21]. Usually matrix semigroups are defined over a field $\mathbb{K}$ [22] (on some nonsupersymmetric generalizations of $\mathbb{K}$-representations see [23,24]). But after discovery of supersymmetry [25,26] the realistic unified particle theories began to be considered in superspace [27,28]. So all variables and functions were defined not over a field $\mathbb{K}$, but over Grassmann-Banach superalgebras over $\mathbb{K}$ [29–31] become in general noninvertible. Therefore they should be considered by semigroup theory, which was claimed in [32,33], some supersymmetric semigroups having nontrivial abstract properties were found in [34], noninvertible extensions of supermanifolds—semisupermanifolds—were introduced in [35–37]. Also, it was shown that supermatrices of the special (antitriangle) shape can form various strange and sandwich semigroups not known before [18,20]. Here we consider one-parametric semigroups (for general theory see [2,5,38]) of antitriangle supermatrices and corresponding superoperator semigroups. The first ones continuously represent idempotent semigroups and second ones lead to new superoperator semigroups with nontrivial properties [21].

2. Preliminaries

Let Λ be a commutative Banach $\mathbb{Z}_2$-graded superalgebra [15,39] over a field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{Q}_p$) with a decomposition into the direct sum: $\Lambda = \Lambda_{\bar{0}} \oplus \Lambda_{\bar{1}}$. The elements a from $\Lambda_{\bar{0}}$ and $\Lambda_{\bar{1}}$ are homogeneous with respect to the parity defined by $\mathsf{p}(a) \stackrel{\text{def}}{=} \left\{ \bar{i} \in \left\{ \bar{0}, \bar{1} \right\} = \mathbb{Z}_2 | \, a \in \Lambda_{\bar{i}} \right\}$. A *supercommutator* is defined by $[a, b] = ab - (-1)^{\mathsf{p}(a)\mathsf{p}(b)} ba$. In the simplest case, if we have the Grassmann algebra $\Lambda(n)$ with generators $\xi_i, \ldots, \xi_n$ satisfying $\xi_i \xi_j + \xi_j \xi_i = 0,\ 1 \leqslant i,\ j \leqslant n,$ in particular $\xi_i^2 = 0$ (n can be infinite), then any even x and odd $\varkappa$ elements [1] have the expansions

$$\begin{aligned} x &= x_{\text{numb}} + x_{\text{nil}} = x_0 + x_{12}\xi_1\xi_2 + x_{13}\xi_1\xi_3 + \cdots \\ &= x_{\text{numb}} + \sum_{1 \leqslant r \leqslant n} \sum_{1 < i_1 < \cdots < i_{2r} \leqslant n} x_{i_1 \cdots i_{2r}} \xi_{i_1} \cdots \xi_{i_{2r}}, \end{aligned} \tag{1}$$

$$\begin{aligned} \varkappa &= \varkappa_{\text{nil}} = x_1\xi_1 + x_2\xi_2 + \cdots + x_{123}\xi_1\xi_2\xi_3 + \cdots \\ &= \sum_{1 \leqslant r \leqslant n} \sum_{1 < i_1 < \cdots < i_{2r-1} \leqslant n} x_{i_1 \cdots i_{2r-1}} \xi_{i_1} \cdots \xi_{i_{2r-1}}, \end{aligned} \tag{2}$$

where $x_{i_1 \cdots i_n} \in \mathbb{K}$. The structure of superalgebra on $\Lambda(n)$ is defined by putting $\mathsf{p}(\xi_i) = \bar{1}$ [16]. The map ε dropping all odd generators is called a number map (canonical projection [17], body map [40]) which acts on the even and objects (1)–(2) as $\varepsilon(x) = x|_{\xi_i = 0} = x_{\text{numb}}$, $\varepsilon(\varkappa) = \varkappa|_{\xi_i = 0} = 0$.

[1] Here and in what follows we will use Latin letter for even objects and Greek letters for odd ones.

From (1)–(2) it follows that e.g., the equations $x^2 = 0$, $\varkappa x = 0$, and $\varkappa\varkappa' = 0$ can have nonzero nontrivial solutions, i.e., zero-divisors and even nilpotents can appear on a par with invertible solutions of various equations. For instance, in $\Lambda(4)$ even nonvanishing nilpotents $x^2 = 0$ satisfy $x_0 = 0$, $x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0$ and for the components of nonvanishing zero-divisors $\varkappa x = 0$ we obtain the relations $x_0 = 0$,

$$x_1x_{23} - x_2x_{13} + x_3x_{12} = 0, \quad x_1x_{24} - x_2x_{14} + x_4x_{12} = 0,$$
$$x_1x_{34} - x_3x_{14} + x_4x_{13} = 0, \quad x_2x_{34} - x_3x_{24} + x_4x_{23} = 0.$$

From $\varkappa\varkappa' = 0$ we get the conditions $x_i x'_j - x_j x'_i = 0$, $i, j = 1, 2, 3, 4$, which obviously shows that odd objects (2) are nilpotents of second degree [2] $\varkappa^2 = 0$.

Claim 1. *If zero divisors and nilpotents will be included in the following analysis as elements of matrices, then one can find new and unusual properties of corresponding semigroups.*

So we consider general properties of supermatrices [15,16] and then introduce their additional nontrivial reductions [18,21].

Let us consider $(p \,|\, q)$-dimensional linear model superspace $\Lambda^{p|q}$ over Λ as the even sector of the direct product $\Lambda^{p|q} = \Lambda^{p}_{\overline{0}} \times \Lambda^{q}_{\overline{1}}$ [30,31,40]. The even morphisms $\mathrm{Hom}_{\overline{0}}(\Lambda^{p\,|\,q}, \Lambda^{m|n})$ between superlinear spaces $\Lambda^{p|q} \to \Lambda^{m|n}$ are described by means of $(m + n) \times (p + q)$-supermatrices [15,16] (for some nontrivial properties see [42, 43]. In what follows we will treat noninvertible morphisms [44,45] on a par with invertible ones [18,21], i.e. we will consider also noninvertible supermatrices which form a general linear semigroup $\mathrm{Mat}_{\Lambda}(p \,|\, q)$[15].

We remind here some necessary facts from supermatrix theory [15–17]. The standard square supermatrix structure $M \in \mathrm{Mat}_{\Lambda}(p \,|\, q)$ can be written in the block shape [3]

$$M = \begin{pmatrix} A_{p\times p} & \Sigma_{p\times q} \\ \Delta_{q\times p} & B_{q\times q} \end{pmatrix}, \tag{3}$$

where ordinary matrices A, B and Σ, Δ (we will drop dimension indices) consist of even (odd) and odd (even) elements for even $\mathsf{p}(M) = \overline{0}$ (odd $\mathsf{p}(M) = \overline{1}$) supermatrix M respectively [16]. For sets (of matrices and other objects below) we use corresponding bold symbols, and the set product is standard $\mathbf{M} \cdot \mathbf{N} \overset{\text{def}}{=} \left\{ \bigcup MN \mid M, N \in \mathrm{Mat}_{\Lambda}(p \,|\, q) \right\}$. The set of invertible supermatrices $\mathbf{M}^{\text{inv}}$ from $\mathrm{Mat}_{\Lambda}(p \,|\, q)$ form the general linear group $\mathrm{GMat}_{\Lambda}(p \,|\, q)$ and for them $\varepsilon(A) \neq 0$, $\varepsilon(B) \neq 0$ [15]. The ideal structure of supermatrices can be described as follows [21]. We introduce the sets

[2] In [41] an invertible odd object θ_0 was introduced to investigate to study supersymmetric pseudodifferential operators.

[3] For nonstandard diagonal formats of supermatrices see [46].

$$\mathbf{M}' = \{M \in \mathrm{Mat}_{\mathbf{\Lambda}}(p|q) \mid \varepsilon(A) \neq 0\}, \tag{4}$$
$$\mathbf{M}'' = \{M \in \mathrm{Mat}_{\mathbf{\Lambda}}(p|q) \mid \varepsilon(B) \neq 0\}, \tag{5}$$
$$\mathbf{J}' = \{M \in \mathrm{Mat}_{\mathbf{\Lambda}}(p|q) \mid \varepsilon(A) = 0\}, \tag{6}$$
$$\mathbf{J}'' = \{M \in \mathrm{Mat}_{\mathbf{\Lambda}}(p|q) \mid \varepsilon(B) = 0\}. \tag{7}$$

Then $\mathbf{M} = \mathbf{M}' \cup \mathbf{J}' = \mathbf{M}'' \cup \mathbf{J}''$ and $\mathbf{M}' \cap \mathbf{J}' = \emptyset$, $\mathbf{M}'' \cap \mathbf{I}'' = \emptyset$, therefore $\mathbf{M}^{\mathrm{inv}} = \mathbf{M}' \cap \mathbf{M}''$. The set $\mathbf{J} = \mathbf{J}' \cap \mathbf{J}''$ is an ideal of the semigroup $\mathfrak{M} = \{\mathbf{M}; \cdot\} = \mathrm{Mat}_{\mathbf{\Lambda}}(p|q)$. Moreover:

(1) Sets $\mathbf{J}$, $\mathbf{J}'$ and $\mathbf{J}''$ are isolated ideals of $\mathrm{Mat}_{\mathbf{\Lambda}}(p|q)$.
(2) Sets $\mathbf{M}^{\mathrm{inv}}$, $\mathbf{M}'$ and $\mathbf{M}''$ are filters of $\mathrm{Mat}_{\mathbf{\Lambda}}(p|q)$.
(3) Sets $\mathbf{M}'$ and $\mathbf{M}''$ (that correspond to $\mathrm{G}'\mathrm{Mat}_{\mathbf{\Lambda}}(p|q)$ and $\mathrm{G}''\mathrm{Mat}(p,q|\Lambda)$ in the Berezin's notation [15]) are subsemigroups [4] of $\mathrm{Mat}_{\mathbf{\Lambda}}(p|q)$, and $\mathbf{M}' = \mathbf{M}^{\mathrm{inv}} \cup \mathbf{J}'$ and $\mathbf{M}'' = \mathbf{M}^{\mathrm{inv}} \cup \mathbf{J}''$, where corresponding isolated ideals are $\mathbf{K}' = \mathbf{M}' \setminus \mathbf{M}^{\mathrm{inv}} = \mathbf{M}' \cap \mathbf{J}''$ and $\mathbf{K}'' = \mathbf{M}'' \setminus \mathbf{M}^{\mathrm{inv}} = \mathbf{M}'' \cap \mathbf{J}'$.
(4) The ideal $\mathbf{J}$ of $\mathrm{Mat}_{\mathbf{\Lambda}}(p|q)$ is $\mathbf{J} = \mathbf{J}' \cup \mathbf{K}' = \mathbf{J}'' \cup \mathbf{K}''$(for details see [20,21]).

The superanalog of trace is defined by [15,16]

$$\mathrm{str}\, M = \mathrm{tr}\, A - \mathrm{tr}\, B, \quad \mathsf{p}(M) = \bar{0}, \tag{8}$$
$$\mathrm{str}\, M = \mathrm{tr}\, A + \mathrm{tr}\, B, \quad \mathsf{p}(M) = \bar{1} \tag{9}$$

and has the properties $\mathrm{str}\,(MN) = (-1)^{\mathsf{p}(M)\mathsf{p}(N)}\, \mathrm{str}\,(NM)$ and $\mathrm{str}\left(UMU^{-1}\right) = \mathrm{str} M$, where $U \in \mathbf{M}^{\mathrm{inv}}$ and the inverse supermatrix U^{-1} is the solution of the equation $UU^{-1} = I$. The superanalog of determinant—Berezinian [15]—is well defined for invertible supermatrices from $\mathbf{M}^{\mathrm{inv}}$ and is determined through ordinary determinants and Shur complements as follows [48–50]

$$\mathrm{Ber} M = \frac{\det\left(A - \Sigma B^{-1} \Delta\right)}{\det B}, \tag{10}$$

$$\frac{1}{\mathrm{Ber} M} = \frac{\det\left(B - \Delta A^{-1} \Sigma\right)}{\det A}. \tag{11}$$

Berezinian is multiplicative $\mathrm{Ber} MN = \mathrm{Ber} M \mathrm{Ber} N$ [15,51] and connected with supertrace by the standard formula $\mathrm{Ber} M = \exp \mathrm{str} \ln M$ [49]. From (4)–(5) and (10)–(11) it follows that $\mathrm{Ber} M$ can be extended to the semigroup $\mathbf{M}''$, while $(\mathrm{Ber} M)^{-1}$ holds valid for the semigroup $\mathbf{M}'$, and their multiplicativity preserves respectively [15] (see the Footnote 4). The analog of Berezinian for the whole semigroup $\mathfrak{M}$ was constructed in [52] for some special cases using theory of generalized inverses

[4] Unfortunately, after translation of the fundamental Berezin's book on supermathematics the russian word "polugruppa/semigroup" denoting $\mathrm{G}'\mathrm{Mat}_{\mathbf{\Lambda}}(p|q)$, $\mathrm{G}''\mathrm{Mat}(p,q|\Lambda)$ (see the original edition [47, s. 89,97]) appeared as "subgroup" (see the translation [15, pp. 95,103]), which perhaps became an obstacle in the way of supermatrix semigroups investigation.

(for ordinary matrices see [53–55]). For instance, if the block B in (3) has a generalized inverse B^+ satisfying $BB^+B = B$, $B^+BB^+ = B^+$ (see [44,56]), then

$$\mathrm{sdet}_{(+)}M = \det\left(A - \Sigma B^+ \Delta\right)\det B^+ \tag{12}$$

can be treated as an analog of superdeterminant for noninvertible supermatrices of this special kind which reproduces the standard formula (10) in the invertible case $B^+ = B^{-1}$ [52].

3. Triangle–antitriangle/even–odd classification of supermatrices

Let us consider $(1+1) \times (1+1)$-supermatrices describing the elements from $\mathrm{Hom}_0(\Lambda^{1|1}, \Lambda^{1|1})$ in the standard $\Lambda^{1|1}$ basis [15]

$$M \equiv \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} \in \mathrm{Mat}_\Lambda(1|1) \tag{13}$$

where $a, b \in \Lambda_{\bar{0}}$, $\alpha, \beta \in \Lambda_{\bar{1}}$, $\alpha^2 = \beta^2 = 0$ (in the following we use Latin letters for elements from $\Lambda_{\bar{0}}$ and Greek letters for ones from $\Lambda_{\bar{1}}$, and odd elements are nilpotent of index 2). The supertrace and Berezinian are defined by [15] ($\varepsilon(b) \neq 0$, which corresponds to $\mathbf{M}''$ (5))

$$\mathrm{str}M = a - b, \tag{14}$$

$$\mathrm{Ber}M = \tfrac{a}{b} + \tfrac{\beta\alpha}{b^2}. \tag{15}$$

First term of (15) is related to triangle supermatrices, second term – to antitriangle ones. So we obviously have different *two dual types* of supermatrices [18].

Definition 1. *Even-reduced supermatrices* are elements from $\mathrm{Mat}_\Lambda(1|1)$ of the form

$$M_{\mathrm{even}} \equiv \begin{pmatrix} a & \alpha \\ 0 & b \end{pmatrix} \in \mathrm{RMat}_\Lambda^{\mathrm{even}}(1|1) \subset \mathrm{Mat}_\Lambda(1|1). \tag{16}$$

Odd-reduced supermatrices are elements from $\mathrm{Mat}_\Lambda(1|1)$ of the form

$$M_{\mathrm{odd}} \equiv \begin{pmatrix} 0 & \alpha \\ \beta & b \end{pmatrix} \in \mathrm{RMat}_\Lambda^{\mathrm{odd}}(1|1) \subset \mathrm{Mat}_\Lambda(1|1). \tag{17}$$

The odd-reduced supermatrices have a nilpotent (but nonzero) Berezinian

$$\mathrm{Ber}M_{\mathrm{odd}} = \frac{\beta\alpha}{b^2} \neq 0, \tag{18}$$

$$(\mathrm{Ber}M_{\mathrm{odd}})^2 = 0. \tag{19}$$

Remark 1. Indeed property (19) prevented in the past the use of this type (odd-reduced) of supermatrices in physics. All previous applications (excluding [18,19,57]) were connected with triangle (even-reduced, similar to Borel) ones and first term in Berezinian $\mathrm{Ber}M = a/b$ (15).

Remark 2. Odd-reduced supermatrices belong to the semigroup $\mathrm{GMat}''(1, 1|\Lambda)$ in Berezin's notations and to $\mathbf{M}''$ (5) (2–2 block is invertible), and for them "superdeterminant preserves multiplicativity" [15].

The odd-reduced supermatrices satisfy

$$M_{\mathrm{odd}}^n = b^{n-2}\begin{pmatrix} \alpha\beta & \alpha b \\ \beta b & b^2 - (n-1)\,\alpha\beta \end{pmatrix}, \tag{20}$$

which gives $\mathrm{Ber} M_{\mathrm{odd}}^n = 0$ and $\mathrm{str}\, M_{\mathrm{odd}}^n = b^{n-2}\left(n\alpha\beta - b^2\right)$.

The even- and odd-reduced supermatrices are *mutually dual* in the sense of the Berezinian addition formula [18]

$$\mathrm{Ber} M = \mathrm{Ber} M_{\mathrm{even}} + \mathrm{Ber} M_{\mathrm{odd}}. \tag{21}$$

The matrices from $\mathrm{Mat}\,(1|1)$ form a linear semigroup $\mathfrak{M}$ of $(1+1)\times(1+1)$-supermatrices under the standard supermatrix multiplication $\mathfrak{M}\,(1|1) \stackrel{\mathrm{def}}{=} \{\mathbf{M}|\cdot\}$ [15]. Obviously, the even-reduced matrices M_{even} form a semigroup $\mathfrak{M}_{\mathrm{even}}\,(1|1)$ which is a subsemigroup of $\mathfrak{M}\,(1|1)$, because of $\mathbf{M}_{\mathrm{even}} \cdot \mathbf{M}_{\mathrm{even}} \subseteq \mathbf{M}_{\mathrm{even}}$ and the unity is in $\mathfrak{M}_{\mathrm{even}}\,(1|1)$. This trivial observation leads to general structure (Borel) theory for matrices: triangle matrices form corresponding substructures (subgroups and subsemigroups). It was believed before that in case of supermatrices the situation does not changed, because supermatrix multiplication is the same [15]. But they did not take into account *zero divisors and nilpotents* appearing naturally and inevitably in supercase.

Claim 2. *Standard* (*lower/upper*) *triangle supermatrices are not the only substructures due to unusual properties of zero divisors and nilpotents appearing among elements* (*see* (1)–(2) *and following examples*).

It means that in such consideration we have additional (to the triangle) class of subsemigroups. Then we can formulate the following general

Problem 1. For a given n, m, p, q to describe and classify all possible substructures (subgroups and subsemigroups) of $(m+n)\times(p+q)$-supermatrices.

First example of such new substructures are $\mathbf{\Gamma}$-matrices considered below.

Claim 3. *These new substructures lead to* new corresponding superoperators *which are represented by one-parameter substructures of supermatrices.*

Therefore we first consider possible (not triangle) subsemigroups of supermatrices.

4. Semigroups of odd-reduced supermatrices

In general, the odd-reduced matrices M_{odd} do not form a semigroup, since their multiplication is not closed in general $\mathbf{M}_{\text{odd}} \cdot \mathbf{M}_{\text{odd}} \subset \mathbf{M}$. Nevertheless, some subset of $\mathbf{M}_{\text{odd}}$ *can form* a semigroup [18,21]. That can happen due to the existence of zero divisors in Λ, and so we have $\mathbf{M}_{\text{odd}} \cdot \mathbf{M}_{\text{odd}} \cap \mathbf{M}_{\text{odd}} = \mathbf{M}_{\text{odd}}^{\text{smg}} \neq \emptyset$.

To find the set $\mathbf{M}_{\text{odd}}^{\text{smg}}$ we consider a $(1+1) \times (1+1)$ example. Let $\alpha, \beta \in \mathbf{\Gamma}$, where $\mathbf{\Gamma} \subset \Lambda_1$. We denote $\operatorname{Ann} \alpha \stackrel{\text{def}}{=} \{\gamma \in \Lambda_1 \mid \gamma \cdot \alpha = 0\}$ and $\operatorname{Ann} \mathbf{\Gamma} = \bigcap_{\alpha \in \mathbf{\Gamma}} \operatorname{Ann} \alpha$ (here the intersection is crucial). Then we define sets of *left* and *right* $\mathbf{\Gamma}$*-matrices*

$$\mathbf{M}_{\text{odd(L)}}^{\mathbf{\Gamma}} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & \mathbf{\Gamma} \\ \operatorname{Ann} \mathbf{\Gamma} & \mathbf{b}_{\text{L}} \end{pmatrix}, \tag{22}$$

$$\mathbf{M}_{\text{odd(R)}}^{\mathbf{\Gamma}} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & \operatorname{Ann} \mathbf{\Gamma} \\ \mathbf{\Gamma} & \mathbf{b}_{\text{R}} \end{pmatrix}, \tag{23}$$

where $\mathbf{b}_{\text{L}}, \mathbf{b}_{\text{R}} \in \Lambda_0$.

Proposition 1. *The left and right* $\mathbf{\Gamma}$*-matrices* $M_{\text{odd(L,R)}}^{\mathbf{\Gamma}} \subset \mathbf{M}_{\text{odd}}$ *form two different subsemigroups* $\mathfrak{M}_{\text{odd(L)}}^{\mathbf{\Gamma}}(1|1)$ *and* $\mathfrak{M}_{\text{odd(R)}}^{\mathbf{\Gamma}}(1|1)$ *of* $\mathfrak{M}(1|1)$ *under the standard supermatrix multiplication, iff* $\mathbf{b}_{\text{L}}\mathbf{\Gamma} \subseteq \mathbf{\Gamma}$ *and* $\mathbf{b}_{\text{R}} \operatorname{Ann} \mathbf{\Gamma} \subseteq \operatorname{Ann} \mathbf{\Gamma}$, *respectively.*

Proof. It follows from the equality

$$\begin{pmatrix} 0 & \gamma_1 \\ \gamma_1' & b_1 \end{pmatrix} \begin{pmatrix} 0 & \gamma_2 \\ \gamma_2' & b_2 \end{pmatrix} = \begin{pmatrix} 0 & \gamma_1 b_2 \\ b_1 \gamma_2' & b_1 b_2 \end{pmatrix}$$

and from definition of $\operatorname{Ann} \mathbf{\Gamma}$ and $\mathbf{\Gamma} \operatorname{Ann} \mathbf{\Gamma} = 0$, and in general $\mathbf{\Gamma} \neq \operatorname{Ann} \mathbf{\Gamma}$. □

Corollary 1. *The introduced antitriangle* $\mathbf{\Gamma}$*-matrices are additional to triangle supermatrices substructures which form subsemigroups of the general linear semigroup of all supermatrices* $\mathfrak{M}$.

Let us consider general square antitriangle $(p+q) \times (p+q)$-supermatrices (having even parity in notations of [15]) of the form

$$M_{\text{odd}}^{p|q} \stackrel{\text{def}}{=} \begin{pmatrix} 0_{p \times p} & \varSigma_{p \times q} \\ \varDelta_{q \times p} & B_{q \times q} \end{pmatrix}, \tag{24}$$

where ordinary matrix $B_{q \times q}$ consists of even elements and matrices $\varSigma_{p \times q}$ and $\varDelta_{q \times p}$ consist of odd elements [15,16] (we drop their indices below). The Berezinian of $M_{\text{odd}}^{p|q}$ can be obtained from the general formula (10) by reduction and in case of invertible B (which is implied here) is (cf. (18))

$$\operatorname{Ber} M_{\text{odd}}^{p|q} = -\frac{\det\left(\varSigma B^{-1} \varDelta\right)}{\det B}. \tag{25}$$

The Berezinian $\operatorname{Ber} M_{\text{odd}}^{p|q}$ is multiplicative (see Remark 2).

Assertion 1. A set of supermatrices $\mathbf{M}_{\mathrm{odd}}^{p|q}$ form a semigroup $\mathfrak{M}_{\mathrm{odd}}^{\mathbf{\Gamma}}(p|q)$ of $\Gamma^{p|q}$-matrices, if $\Sigma\mathbf{\Delta} = 0$, i.e. antidiagonal matrices are orthogonal, and $\mathbf{\Sigma B} \subset \mathbf{\Sigma}$, $\mathbf{B\Delta} \subset \mathbf{\Delta}$.

Proof. Consider the product

$$M_{\mathrm{odd}_1}^{p|q} M_{\mathrm{odd}_2}^{p|q} = \begin{pmatrix} \Sigma_1 \Delta_2 & \Sigma_1 B_2 \\ B_1 \Delta_2 & B_1 B_2 + \Delta_1 \Sigma_2 \end{pmatrix} \tag{26}$$

and observe the condition of vanishing even-even block, which gives $\Sigma_1 \Delta_2 = 0$, and other conditions follows obviously. □

From (26) it follows

Corollary 2. *Two $\Gamma^{p|q}$-matrices satisfy the band relation $M_1 M_2 = M_1$, if $\Sigma_1 B_2 = \Sigma_1,\ B_1 \Delta_2 = \Delta_2,\ B_1 B_2 + \Delta_1 \Sigma_2 = B_1$.*

Definition 2. We call a set of $\Gamma^{p|q}$-matrices satisfying additional condition $\mathbf{\Delta\Sigma} = 0$, a set of strong $\Gamma^{p|q}$-matrices.

Strong $\Gamma^{p|q}$-matrices have some extra nice features and all supermatrices considered below are of this class.

Corollary 3. *Idempotent strong $\Gamma^{p|q}$-matrices are defined by relations $\Sigma B = \Sigma$, $B\Delta = \Delta,\ B^2 = B$.*

The product of n strong $\Gamma^{p|q}$-matrices M_i has the following form

$$M_1 M_2 \cdots M_n = \begin{pmatrix} 0 & \Sigma_1 A_{n-1} B_n \\ B_1 A_{n-1} \Delta_n & B_1 A_{n-1} B_n \end{pmatrix}, \tag{27}$$

where $A_{n-1} = B_2 B_3 \cdots B_{n-1}$, and its Berezinian is

$$\mathrm{Ber}\,(M_1 M_2 \cdots M_n) = -\frac{\det\,(\Sigma_1 A_{n-1} \Delta_n)}{\det\,(B_1 A_{n-1} B_n)}. \tag{28}$$

5. Idempotent semigroups of one-even-parameter supermatrices

Here we investigate one-even-parameter subsemigroups of $\mathbf{\Gamma}$-semigroups and as a particular example for clearness of statements consider $\mathfrak{M}_{\mathrm{odd}}(1|1)$, where all characteristic features taking place in general $(p+q) \times (p+q)$ as well can be seen. These formulas will be applied for establishing corresponding superoperator semigroup properties.

A simplest semigroup can be constructed from antidiagonal nilpotent supermatrices of the shape

$$Y_\alpha(t) \stackrel{\text{def}}{=} \begin{pmatrix} 0 & \alpha t \\ \alpha & 0 \end{pmatrix}. \tag{29}$$

where $t \in \Lambda^{1|0}$ is an even parameter of the Grassmann algebra Λ which continuously parametrizes elements $Y_\alpha(t)$ and $\alpha \in \Lambda^{0|1}$ is a fixed nilpotent odd element of Λ ($\alpha^2 = 0$) which labels the sets $\mathbf{Y}_\alpha = \bigcup_t Y_\alpha(t)$.

Definition 3. The supermatrices $Y_\alpha(t)$ together with a null supermatrix

$$Z \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

form a *continuous null semigroup* $\mathfrak{Z}_\alpha(1|1) = \{\mathbf{Y}_\alpha \cup Z; \cdot\}$ having the null multiplication

$$Y_\alpha(t)\, Y_\alpha(u) = Z. \tag{30}$$

Assertion 2. For any fixed $t \in \Lambda^{1|0}$ the set $\{Y_\alpha(t), Z\}$ is a 0-minimal ideal in $\mathfrak{Z}_\alpha(1|1)$.

Remark 3. If we consider, for instance, a one-even-parameter odd-reduced supermatrix

$$R_\alpha(t) = \begin{pmatrix} 0 & \alpha \\ \alpha & t \end{pmatrix},$$

then multiplication of $R_\alpha(t)$ is not closed since

$$R_\alpha(t)\, R_\alpha(u) = \begin{pmatrix} 0 & \alpha u \\ \alpha t & tu \end{pmatrix} \notin \mathbf{R}_\alpha = \bigcup_t R_\alpha(t).$$

Any other possibility except ones considered below also do not give closure of multiplication.

Thus the only nontrivial closed systems of one-even-parameter odd-reduced (antitriangle) $(1+1) \times (1+1)$ supermatrices are $\mathbf{P}_\alpha = \bigcup_t P_\alpha(t)$, where

$$P_\alpha(t) \stackrel{\text{def}}{=} \begin{pmatrix} 0 & \alpha t \\ \alpha & 1 \end{pmatrix} \tag{31}$$

and $\mathbf{Q}_\alpha = \bigcup_t Q_\alpha(u)$ where

$$Q_\alpha(u) \stackrel{\text{def}}{=} \begin{pmatrix} 0 & \alpha \\ \alpha u & 1 \end{pmatrix}. \tag{32}$$

First, we establish multiplication properties of supermatrices $P_\alpha(t)$ and $Q_\alpha(u)$. Obviously, that they are idempotent.

Assertion 3. Sets of idempotent supermatrices $\mathbf{P}_\alpha$ and $\mathbf{Q}_\alpha$ form left zero and right zero semigroups respectively with multiplication

$$P_\alpha(t)\,P_\alpha(u) = P_\alpha(t)\,, \tag{33}$$
$$Q_\alpha(t)\,Q_\alpha(u) = Q_\alpha(u)\,. \tag{34}$$

Proof. It simply follows from supermatrix multiplication law and nilpotence of α. □

Corollary 4. *The sets $\mathbf{P}_\alpha$ and $\mathbf{Q}_\alpha$ are rectangular bands since*

$$P_\alpha(t)\,P_\alpha(u)\,P_\alpha(t) = P_\alpha(t)\,, \tag{35}$$
$$P_\alpha(u)\,P_\alpha(t)\,P_\alpha(u) = P_\alpha(u) \tag{36}$$

and

$$Q_\alpha(u)\,Q_\alpha(t)\,Q_\alpha(u) = Q_\alpha(u)\,, \tag{37}$$
$$Q_\alpha(t)\,Q_\alpha(u)\,Q_\alpha(t) = Q_\alpha(t) \tag{38}$$

with components $t = t_0 + \operatorname{Ann}\alpha$ and $u = u_0 + \operatorname{Ann}\alpha$ correspondingly.

They are orthogonal in sense of

$$Q_\alpha(t)P_\alpha(u) = E_\alpha, \tag{39}$$

where

$$E_\alpha \stackrel{\text{def}}{=} \begin{pmatrix} 0 & \alpha \\ \alpha & 1 \end{pmatrix} \tag{40}$$

is a right "unity" and left "zero in semigroup $\mathbf{P}_\alpha$, because

$$P_\alpha(t)\,E_\alpha = P_\alpha(t)\,, \quad E_\alpha P_\alpha(t) = E_\alpha \tag{41}$$

and a left "unity" and right "zero" in semigroup $\mathbf{Q}_\alpha$, because

$$Q_\alpha(t)\,E_\alpha = E_\alpha, \quad E_\alpha Q_\alpha(t) = Q_\alpha(t)\,. \tag{42}$$

It is important to note that

$$P_\alpha(t=1) = Q_\alpha(t=1) = E_\alpha, \tag{43}$$

and so $\mathbf{P}_\alpha \cap \mathbf{Q}_\alpha = E_\alpha$. Therefore, almost all properties of $\mathbf{P}_\alpha$ and $\mathbf{Q}_\alpha$ are similar, and we will consider only one of them in what follows. For generalized Green's relations and more detail properties of odd-reduced supermatrices see [19,20].

6. Odd-reduced supermatrix operator semigroups

Let us consider a semigroup $\mathscr{P}$ of superoperators $\mathsf{P}(t)$ (see for general theory [2,3,5,58]) represented by the one-even-parameter semigroup $\mathbf{P}_\alpha$ of odd-reduced supermatrices $P_\alpha(t)$ (31) which act on $(1|1)$-dimensional superspace $\mathbb{R}^{1|1}$ as follows

$P_\alpha(t)\mathtt{X}$, where $\mathtt{X} = \binom{x}{\varkappa} \in \mathbb{R}^{1|1}$, where x is an even coordinate, $\varkappa$ is an odd coordinate $\left(\varkappa^2 = 0\right)$ having expansions (1) and (2). We have a representation $\rho : \mathscr{P} \to \mathbf{P}_\alpha$ with correspondence $\mathsf{P}(t) \to P_\alpha(t)$, but (as is usually made, e.g., [5]) we identify space of superoperators with the space of corresponding matrices (nevertheless, we use here operator notations for convenience).

Definition 4. An odd-reduced "dynamical" system on $\mathbb{R}^{1|1}$ is defined by an odd-reduced supermatrix-valued function $\mathsf{P}(\cdot) : \mathbb{R}_+ \to \mathfrak{M}_{\text{odd}}(1|1)$ and "time evolution" of the state $\mathtt{X}(0) \in \mathbb{R}^{1|1}$ given by the function $\mathtt{X}(t) : \mathbb{R}_+ \to \mathbb{R}^{1|1}$, where

$$\mathtt{X}(t) = \mathsf{P}(t)\,\mathtt{X}(0) \tag{44}$$

and can be called as orbit of $\mathtt{X}(0)$ under $\mathsf{P}(\cdot)$.

Remark 4. In general the definition, the continuity, the functional equation and most of conclusions below hold valid also for $t \in \mathbb{R}^{1|0}$ (as e.g. in [5, p. 9]) including "nilpotent time" directions (see expansions (1) and (2)).

From (33) it follows that

$$\mathsf{P}(t)\,\mathsf{P}(s) = \mathsf{P}(t), \tag{45}$$

and so superoperators $\mathsf{P}(t)$ are idempotent. Also they form a rectangular band, because of

$$\mathsf{P}(t)\mathsf{P}(s)\mathsf{P}(t) = \mathsf{P}(t), \tag{46}$$

$$\mathsf{P}(s)\mathsf{P}(t)\mathsf{P}(s) = \mathsf{P}(s). \tag{47}$$

We observe that

$$\mathsf{P}(0) = \begin{pmatrix} 0 & 0 \\ \alpha & 1 \end{pmatrix} \neq \mathsf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{48}$$

as opposite to the standard case [2]. A "generator" $\mathsf{A} = \mathsf{P}'(t)$ is

$$\mathsf{A} = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \tag{49}$$

and so the standard definition of generator [2]

$$\mathsf{A} = \lim_{t \to 0} \frac{\mathsf{P}(t) - \mathsf{P}(0)}{t}. \tag{50}$$

holds and for difference we have the standard relation

$$\mathsf{P}(t) - \mathsf{P}(s) = \mathsf{A} \cdot (t - s). \tag{51}$$

The following properties of the generator A take place

$$\mathsf{P}(t)\,\mathsf{A} = \mathsf{Z}, \tag{52}$$

$$\mathsf{A}\mathsf{P}(t) = \mathsf{A}, \tag{53}$$

where "zero operator" Z is represented by the null supermatrix, $\mathsf{A}^2 = \mathsf{Z}$, and therefore generator A is a nilpotent of second degree.

From (50) it follows that

$$\mathsf{P}(t) = \mathsf{P}(0) + \mathsf{A} \cdot t. \tag{54}$$

Definition 5. We call operators which can be presented as a linear supermatrix function of t a t-linear superoperators.

From (54) it follows that $\mathsf{P}(t)$ is a t-linear superoperator.

Proposition 2. *Superoperators* $\mathsf{P}(t)$ cannot be presented *as an exponent* (*as for the standard superoperator semigroups* $\mathsf{T}(t) = \mathrm{e}^{\mathsf{A} \cdot t}$ [2]).

Proof. In our case

$$\mathsf{T}(t) = \mathrm{e}^{\mathsf{A} \cdot t} = \mathsf{I} + \mathsf{A} \cdot t = \begin{pmatrix} 1 & \alpha t \\ 0 & 1 \end{pmatrix} \notin \mathbf{P}_\alpha. \qquad \square \tag{55}$$

Remark 5. Exponential superoperator $\mathsf{T}(t) = \mathrm{e}^{\mathsf{A} \cdot t}$ is represented by even-reduced supermatrices $\mathsf{T}(\cdot) : \mathbb{R}_+ \to \mathfrak{M}_{\text{even}}(1|1)$ [5], but idempotent superoperator $\mathsf{P}(t)$ is represented by odd-reduced supermatrices $\mathsf{P}(\cdot) : \mathbb{R}_+ \to \mathfrak{M}_{\text{odd}}(1|1)$ (see Definition 1).

Nevertheless, the superoperator $\mathsf{P}(t)$ satisfies the same linear differential equation

$$\mathsf{P}'(t) = \mathsf{A} \cdot \mathsf{P}(t) \tag{56}$$

as the standard exponential superoperator $\mathsf{T}(t)$ (the initial value problem [5])

$$\mathsf{T}'(t) = \mathsf{A} \cdot \mathsf{T}(t). \tag{57}$$

That leads to the following:

Corollary 5. *In case initial state does not equal unity* $\mathsf{P}(0) \neq \mathsf{I}$, *there exists an additional class of solutions of the initial value problem* (56)–(57) *among odd-reduced* (*antidiagonal*) *idempotent t-linear* (*nonexponential*) *superoperators*.

Let us compare behavior of superoperators $\mathsf{P}(t)$ and $\mathsf{T}(t)$. First of all, their generators coincide

$$\mathsf{P}'(0) = \mathsf{T}'(0) = \mathsf{A}. \tag{58}$$

But powers of $\mathsf{P}(t)$ and $\mathsf{T}(t)$ are different $\mathsf{P}^n(t) = \mathsf{P}(t)$ and $\mathsf{T}^n(t) = \mathsf{T}(nt)$. In their common actions the superoperator which is from the left transfers its properties to the right hand side as follows

$$\mathsf{T}^n(t)\mathsf{P}(t) = \mathsf{P}((n+1)t), \tag{59}$$

$$\mathsf{P}^n(t)\mathsf{T}(t) = \mathsf{P}(t). \tag{60}$$

Their commutator is nonvanishing

$$\left[\mathsf{T}\,(t)\,\mathsf{P}\,(s)\right] = \mathsf{P}'\,(0)\,t = \mathsf{T}'\,(0)\,t = \mathsf{A}t, \tag{61}$$

which can be compared with the pure exponential commutator (for our case) $\left[\mathsf{T}\,(t)\,\mathsf{T}\,(u)\right] = 0$ and idempotent commutator

$$\left[\mathsf{P}\,(t)\,\mathsf{P}\,(s)\right] = \mathsf{P}'\,(0)\,(t-s) = \mathsf{A}\,(t-s)\,. \tag{62}$$

Assertion 4. All superoperators $\mathsf{P}\,(t)$ and $\mathsf{T}\,(t)$ commute in case of "nilpotent time" and

$$t \in \operatorname{Ann}\alpha. \tag{63}$$

Remark 6. The uniqueness theorem [5, p. 3] holds only for $\mathsf{T}\,(t)$, because the nonvanishing commutator $\left[\mathsf{A}, \mathsf{P}\,(t)\right] = \mathsf{A} \neq 0$.

Corollary 6. *The superoperator* $\mathsf{T}\,(t)$ *is an inner inverse for* $\mathsf{P}\,(t)$, *because of*

$$\mathsf{P}\,(t)\,\mathsf{T}\,(t)\,\mathsf{P}\,(t) = \mathsf{P}\,(t)\,, \tag{64}$$

but it is not an outer inverse, *because*

$$\mathsf{T}\,(t)\,\mathsf{P}\,(t)\,\mathsf{T}\,(t) = \mathsf{P}\,(2t). \tag{65}$$

Let us try to find a (possibly noninvertible) operator U which connects exponential and idempotent superoperators $\mathsf{P}\,(t)$ and $\mathsf{T}\,(t)$.

Assertion 5. The "semi-similarity" relation

$$\mathsf{T}\,(t)\,\mathsf{U} = \mathsf{U}P\,(t) \tag{66}$$

holds if

$$\mathsf{U} = \begin{pmatrix} \sigma\alpha & \sigma \\ 0 & \rho\alpha \end{pmatrix} \tag{67}$$

which is noninvertible triangle and depends from two odd constants, and the "adjoint" relation

$$\mathsf{U}^{*}\mathsf{T}\,(t) = \mathsf{P}\,(t)\,\mathsf{U}^{*} \tag{68}$$

holds if

$$\mathsf{U}^{*} = \begin{pmatrix} 0 & \alpha v t \\ \alpha u & v \end{pmatrix} \tag{69}$$

which is also noninvertible antitriangle and depends from two even constants and "time".

Note that U is nilpotent of third degree, since $\mathsf{U}^2 = \sigma\rho\mathsf{A}$, but the "adjoint" superoperator is not nilpotent at all if v is not nilpotent.

Both A and Z behave as zeroes, but $\mathsf{Y}(t)$ (see (29)) is a two-sided zero for $\mathsf{T}(t)$ only, since

$$\mathsf{T}(t)\,\mathsf{Y}(t) = \mathsf{Y}(t)\,\mathsf{T}(t) = \mathsf{Y}(t)\,, \tag{70}$$

but

$$\mathsf{P}(t)\,\mathsf{Y}(t) = \mathsf{Y}(0)\,, \tag{71}$$

$$\mathsf{Y}(t)\,\mathsf{P}(t) = \mathsf{A}t. \tag{72}$$

If we add A and Z to superoperators $\mathsf{P}(t)$, then we obtain an extended odd-reduced noncommutative superoperator semigroup $\mathscr{P}_{\text{odd}} = \bigcup \mathsf{P}(t) \bigcup \mathsf{A} \bigcup \mathsf{Z}$ with the following Cayley table (for convenience we add $\mathsf{Y}(t)$ and $\mathsf{T}(t)$ as well)

$1 \setminus 2$	$\mathsf{P}(t)$	$\mathsf{P}(s)$	A	Z	$\mathsf{Y}(t)$	$\mathsf{T}(t)$	$\mathsf{T}(s)$
$\mathsf{P}(t)$	$\mathsf{P}(t)$	$\mathsf{P}(t)$	Z	Z	$\mathsf{P}(t)$	$\mathsf{P}(t)$	$\mathsf{P}(t)$
$\mathsf{P}(s)$	$\mathsf{P}(s)$	$\mathsf{P}(s)$	Z	Z	$\mathsf{P}(s)$	$\mathsf{P}(s)$	$\mathsf{P}(s)$
A	A	A	Z	Z	Z	A	A
Z	Z	Z	Z	Z	Z	Z	Z
$\mathsf{Y}(t)$	$\mathsf{A}t$	$\mathsf{A}s$	Z	Z	Z	$\mathsf{Y}(t)$	$\mathsf{Y}(t)$
$\mathsf{T}(t)$	$\mathsf{P}(2t)$	$\mathsf{P}(t+s)$	A	Z	$\mathsf{Y}(t)$	$\mathsf{T}(2t)$	$\mathsf{T}(t+s)$
$\mathsf{T}(s)$	$\mathsf{P}(t+s)$	$\mathsf{P}(2s)$	A	Z	$\mathsf{Y}(t)$	$\mathsf{T}(t+s)$	$\mathsf{T}(2s)$

(73)

It is easily seen that associativity in the left upper square holds, and so the table (73) is actually represents a semigroup of superoperators $\mathscr{P}_{\text{odd}}$ (under supermatrix multiplication).

The analogs of the "smoothing operator" $\mathsf{V}(t)$ [5] are

$$\mathsf{V}_P(t) = \int_0^t \mathsf{P}(s)\,\mathrm{d}s = \frac{t}{2}\left(\mathsf{P}(t) + \mathsf{P}(0)\right) = \begin{pmatrix} 0 & \alpha\frac{t^2}{2} \\ \alpha t & t \end{pmatrix}, \tag{74}$$

$$\mathsf{V}_T(t) = \int_0^t \mathsf{T}(s)\,\mathrm{d}s = \frac{t}{2}\left(\mathsf{T}(t) + \mathsf{T}(0)\right) = \begin{pmatrix} t & \alpha\frac{t^2}{2} \\ 0 & t \end{pmatrix}. \tag{75}$$

Let us consider the differential sequence of sets of superoperators $\mathsf{P}(t)$

$$\mathsf{S}_n \xrightarrow{\partial} \mathsf{S}_{n-1} \xrightarrow{\partial} \cdots \mathsf{S}_1 \xrightarrow{\partial} \mathsf{S}_0 \xrightarrow{\partial} \mathsf{A} \xrightarrow{\partial} \mathsf{Z}, \tag{76}$$

where $\partial = \mathrm{d}/\mathrm{d}t$ and

$$\mathsf{S}_n = \bigcup_t \frac{t^n}{n\,(n-1)\cdots 1}\mathsf{P}\left(\frac{t}{n+1}\right), \tag{77}$$

and by definition

$$\mathsf{S}_0 = \bigcup_t \mathsf{P}(t)\,, \tag{78}$$

$$\mathsf{S}_1 = \bigcup_t \mathsf{V}_P(t) . \tag{79}$$

Now we construct an analog of the standard operator semigroup functional equation [2,5]

$$\mathsf{T}(t+s) = \mathsf{T}(t)\,\mathsf{T}(s) . \tag{80}$$

Using the multiplication law (45) and manifest representation (31). for the idempotent superoperators $\mathsf{P}(t)$ we can formulate

Definition 6. The odd-reduced idempotent superoperators $\mathsf{P}(t)$ satisfy the following generalized functional equation

$$\mathsf{P}(t+s) = \mathsf{P}(t)\,\mathsf{P}(s) + \mathsf{N}(t,s) , \tag{81}$$

where

$$\mathsf{N}(t,s) = \mathsf{P}'(t)\,s.$$

The presence of second term $\mathsf{N}(t,s)$ in the right hand side of the generalized functional equation (81) can be connected with nonautonomous and deterministic properties of systems describing by it [5]. Indeed, from (44) it follows that

$$\begin{aligned} \mathbb{X}(t+s) &= \mathsf{P}(t+s)\,\mathbb{X}(0) = \mathsf{P}(t)\,\mathsf{P}(s)\,\mathbb{X}(0) + \mathsf{P}'(t)\,s\mathbb{X}(0) \\ &= \mathsf{P}(t)\,\mathbb{X}(s) + \mathsf{P}'(t)\,s\mathbb{X}(0) \neq \mathsf{P}(t)\,\mathbb{X}(s) \end{aligned} \tag{82}$$

as opposite to the always implied relation for exponential superoperators $\mathsf{T}(t)$ (translational property [2,5])

$$\mathsf{T}(t)\,\mathbb{X}(s) = \mathbb{X}(t+s) , \tag{83}$$

which follows from (80). Instead of (83), using the band property (45) we obtain

$$\mathsf{P}(t)\,\mathbb{X}(s) = \mathbb{X}(t) , \tag{84}$$

which can be called the "moving time" property.

Problem 2. Find a "dynamical system" with time evolution satisfying the "moving time" property (84) instead of the translational property (83).

Assertion 6. For "nilpotent time" satisfying (63) the generalized functional equation (81) coincides with the standard functional equation (80), and therefore the idempotent operators $\mathsf{P}(t)$ describe autonomous and deterministic "dynamical" system and satisfy the translational property (83).

Proof. Follows from (63) and (82). □

Problem 3. Find all maps $\mathsf{P}(\cdot) : \mathbb{R}_+ \to \mathfrak{M}(p|q)$ satisfying the generalized functional equation (81).

We turn to this problem later, and now consider some features of the Cauchy problem for idempotent superoperators.

7. Additional nonexponential solution of the Cauchy problem

Let us consider an action (44) of superoperator $\mathsf{P}(t)$ in superspace $\mathbb{R}^{1|1}$ as $\mathtt{X}(t) = \mathsf{P}(t)\,\mathtt{X}(0)$, where the initial components are

$$\mathtt{X}(0) = \begin{pmatrix} x_0 \\ \varkappa_0 \end{pmatrix}.$$

From (44) the evolution of the components has the form

$$\begin{pmatrix} x(t) \\ \varkappa(t) \end{pmatrix} = \begin{pmatrix} \alpha\varkappa_0 t \\ \alpha x_0 + \varkappa_0 \end{pmatrix} \tag{85}$$

which shows that superoperator $\mathsf{P}(t)$ does not lead to time dependence of odd components. Then from (85) we see that

$$\mathtt{X}'(t) = \begin{pmatrix} \alpha\varkappa_0 \\ 0 \end{pmatrix} = \text{const.} \tag{86}$$

This is in full agreement with an analog of the Cauchy problem for our case

$$\mathtt{X}'(t) = \mathsf{A}\cdot\mathtt{X}(t)\,. \tag{87}$$

Assertion 7. The solution of the Cauchy problem (87) is given by (44), but the idempotent superoperator $\mathsf{P}(t)$ *cannot be presented* in exponential form as in the standard case [2], but only in the t-linear form $\mathsf{P}(t) = \mathsf{P}(0) + \mathsf{A}\cdot t \neq \mathrm{e}^{\mathsf{A}\cdot t}$, as we have already shown in (54).

This allows us to formulate

Theorem 1. *In superspace the solution of the Cauchy initial problem with the same generator* A *is two-fold and is given by two different type of superoperators*:

1. *Exponential superoperator* $\mathsf{T}(t)$ *represented by the even-reduced supermatrices*;
2. *Idempotent t-linear superoperator* $\mathsf{P}(t)$ *represented by the odd-reduced supermatrices*.

For comparison the standard solution of the Cauchy problem (87)

$$\mathtt{X}(t) = \mathsf{T}(t)\mathtt{X}(0)$$

in components is

$$\begin{pmatrix} x(t) \\ \varkappa(t) \end{pmatrix} = \begin{pmatrix} x_0 + \alpha\varkappa_0 t \\ \varkappa_0 \end{pmatrix}, \tag{88}$$

which shows that the time evolution of even coordinate is also in nilpotent even direction $\alpha\varkappa_0$ as in (85), but with addition of initial (possibly nonilpotent) x_0, while odd coordinate is (another) constant as well. That leads to

Assertion 8. "Even" and "odd" evolutions coincide if even initial coordinate vanishes $x_0 = 0$ or common starting point is pure odd

$$\mathtt{X}(0) = \begin{pmatrix} 0 \\ \varkappa_0 \end{pmatrix}.$$

A very much important formula is the condition of commutativity [2]

$$\left[\mathsf{A}, \mathsf{P}(t)\right] \mathtt{X}(t) = \mathsf{A}\mathtt{X}(t) = \begin{pmatrix} \alpha\varkappa(t) \\ 0 \end{pmatrix} = 0, \tag{89}$$

which satisfies, when $\alpha \cdot \varkappa(t) = 0$, while in the standard case the commutator $\left[\mathsf{A}, \mathsf{T}(t)\right] \mathtt{X}(t) = 0$, i.e. vanishes without any additional conditions [2].

8. Resolvents of exponential and idempotent superoperators

For resolvents $\mathsf{R}_P(z)$ and $\mathsf{R}_T(z)$ we use analog the standard formula from [2] in the form

$$\mathsf{R}_P(z) = \int_0^\infty \mathrm{e}^{-zt} \mathsf{P}(t)\, \mathrm{d}t, \tag{90}$$

$$\mathsf{R}_T(z) = \int_0^\infty \mathrm{e}^{-zt} \mathsf{T}(t)\, \mathrm{d}t. \tag{91}$$

Using the supermatrix representation (31) we obtain

$$\mathsf{R}_P(z) = \begin{pmatrix} 0 & \frac{\alpha}{z^2} \\ \frac{\alpha}{z} & \frac{1}{z} \end{pmatrix}, \tag{92}$$

$$\mathsf{R}_T(z) = \begin{pmatrix} \frac{1}{z} & \frac{\alpha}{z^2} \\ 0 & \frac{1}{z} \end{pmatrix}. \tag{93}$$

We observe, that $\mathsf{R}_T(z)$ satisfies the standard resolvent relation [5]

$$\mathsf{R}_T(z) - \mathsf{R}_T(w) = (w - z)\, \mathsf{R}_T(z)\, \mathsf{R}_T(w), \tag{94}$$

but its analog for $\mathsf{R}_P(z)$

$$\mathsf{R}_P(z) - \mathsf{R}_P(w) = (w - z)\, \mathsf{R}_P(z)\, \mathsf{R}_P(w) + \frac{w - z}{zw^2}\mathsf{A} \tag{95}$$

has additional term proportional to the generator A.

9. Idempotent *t*-linear operators

Here we consider properties of general t-linear (super)operators of the form

$$\mathsf{K}(t) = \mathsf{K}_0 + \mathsf{K}_1 t, \tag{96}$$

where $\mathsf{K}_0 = \mathsf{K}(0)$ and $\mathsf{K}_1 = \mathsf{K}'(0)$ are constant (super)operators represented by $(n \times n)$ matrices or $(p+q) \times (p+q)$ supermatrices with t ("time") independent entries. Obviously, that the generator of a general t-linear (super)operator is

$$\mathsf{A}_K = \mathsf{K}'(0) = \mathsf{K}_1. \tag{97}$$

We will find system of equations for K_0 and K_1 for some special cases appeared in above consideration.

Assertion 9. If a t-linear (super)operator $\mathsf{K}(t)$ satisfies the band equation (45)

$$\mathsf{K}(t)\,\mathsf{K}(s) = \mathsf{K}(t), \tag{98}$$

then it is idempotent and the constant component (super)operators K_0 and K_1 satisfy the system of equations

$$\mathsf{K}_0^2 = \mathsf{K}_0, \tag{99}$$
$$\mathsf{K}_1^2 = \mathsf{Z}, \tag{100}$$
$$\mathsf{K}_1\mathsf{K}_0 = \mathsf{K}_1, \tag{101}$$
$$\mathsf{K}_0\mathsf{K}_1 = \mathsf{Z}, \tag{102}$$

from which it follows, that K_0 is idempotent, K_1 is nilpotent, and K_1 is right divisor of zero and left zero for K_0.

For non-supersymmetric operators we have

Corollary 7. *The components of t-linear operator $\mathsf{K}(t)$ have the following properties: idempotent matrix K_0 is similar to an upper triangular matrix with* 1 *on the main diagonal and nilpotent matrix K_1 is similar to an upper triangular matrix with* 0 *on the main diagonal* [22,59].

Comparing with the previous particular super case (54) we have $\mathsf{K}_0 = \mathsf{P}(0)$ and $\mathsf{K}_1 = \mathsf{A} = \mathsf{P}'(0)$.

Remark 7. In case of $(p+q) \times (p+q)$ supermatrices the triangularization properties of Corollary 7 do not hold valid due to presence divisors of zero and nilpotents among entries (see expansions (1) and (2)), and so the inner structure of the component supermatrices satisfying (99)–(102) can be much different from the standard non-supersymmetric case [22,59].

Let us consider the structure of t-linear operator $\mathsf{K}(t)$ satisfying the generalized functional equation (81).

Assertion 10. If a t-linear (super)operator $\mathsf{K}(t)$ satisfies the generalized functional equation

$$\mathsf{K}(t+s) = \mathsf{K}(t)\,\mathsf{K}(s) + \mathsf{K}'(t)\,s, \tag{103}$$

then its component (super)operators K_0 and K_1 satisfy the system of equations

$$\mathsf{K}_0^2 = \mathsf{K}_0, \tag{104}$$

$$\mathsf{K}_1^2 = \mathsf{Z}, \tag{105}$$

$$\mathsf{K}_1\mathsf{K}_0 = \mathsf{K}_1, \tag{106}$$

$$\mathsf{K}_0\mathsf{K}_1 = \mathsf{Z}. \tag{107}$$

We observe that the systems (99)–(102) and (104)–(107) are fully identical. It is important to observe the connection of the above properties with the differential equation for t-linear (super)operator $\mathsf{K}(t)$

$$\mathsf{K}'(t) = \mathsf{A}_K \cdot \mathsf{K}(t)\,. \tag{108}$$

Using (97) we obtain the equation for components

$$\mathsf{K}_1^2 = \mathsf{Z}, \tag{109}$$

$$\mathsf{K}_1\mathsf{K}_0 = \mathsf{K}_1. \tag{110}$$

That leads to the following

Theorem 2. *For any t-linear* (*super*)*operator $\mathsf{K}(t) = \mathsf{K}_0 + \mathsf{K}_1 t$ the next statements are equivalent*:

(1) $\mathsf{K}(t)$ *is idempotent and satisfies the band equation* (98);
(2) $\mathsf{K}(t)$ *satisfies the generalized functional equation* (103);
(3) $\mathsf{K}(t)$ *satisfies the differential equation* (108) *and has idempotent time independent part $\mathsf{K}_0^2 = \mathsf{K}_0$ which is orthogonal to its generator $\mathsf{K}_0\mathsf{A} = \mathsf{Z}$.*

10. General t-power-type idempotent operators

Let us consider idempotent (super)operators which depend from time by power-type function, and so they have the form

$$\mathsf{K}(t) = \sum_{m=0}^{n} \mathsf{K}_m t^m, \tag{111}$$

where K_m are t-independent (super)operators represented by $(n \times n)$ matrices or $(p+q)\times(p+q)$ supermatrices. This power-type dependence of is very much important for super case, when supermatrix elements take value in Grassmann algebra, and therefore can be nilpotent (see (1)–(2)).

We now start from the band property $\mathsf{K}(t)\,\mathsf{K}(s) = \mathsf{K}(t)$ and then find analogs of the functional equation and differential equation for them. Expanding the band property (98) in component we obtain n-dimensional analog of (99)–(102) as

$$\mathsf{K}_0^2 = \mathsf{K}_0, \tag{112}$$

$$\mathsf{K}_i^2 = \mathsf{Z},\ \ 1 \leqslant i \leqslant n, \tag{113}$$

$$\mathsf{K}_i\mathsf{K}_0 = \mathsf{K}_i,\ \ 1 \leqslant i \leqslant n, \tag{114}$$

$$\mathsf{K}_0\mathsf{K}_i = \mathsf{Z},\ \ 1 \leqslant i \leqslant n, \tag{115}$$

$$\mathsf{K}_i\mathsf{K}_j = \mathsf{Z},\ \ 1 \leqslant i, j \leqslant n, i \neq j. \tag{116}$$

Proposition 3. *The n-generalized functional equation for any t-power-type idempotent* (*super*)*operators* (111) *has the form*

$$\mathsf{K}(t+s) = \mathsf{K}(t)\,\mathsf{K}(s) + \mathsf{N}_n(t,s), \tag{117}$$

where

$$\mathsf{N}_n(t,s) = \sum_{m=1}^{n}\sum_{l=m}^{n}\mathsf{K}_l\frac{l(l-1)\cdots(l-m+1)}{m!}s^m t^{l-m}. \tag{118}$$

Proof. For the difference using the band property (98) we have $\mathsf{N}_n(t,s) = \mathsf{K}(t+s) - \mathsf{K}(t)\,\mathsf{K}(s) = \mathsf{K}(t+s) - \mathsf{K}(t)$. Then we expand in Taylor series around t and obtain $\mathsf{N}_n(t,s) = \sum_{m=1}^{n}\mathsf{K}^{(m)}(t)\,s^m/m!$, where $\mathsf{K}^{(m)}(t)$ denotes nth derivative which is a finite series for the power-type $\mathsf{K}(t)$ (111). □

The differential equation for idempotent (super)operators coincide with the standard initial value problem only for t-linear operators. In case of the power-type operators (111) we have

Proposition 4. *The n-generalized differential equation for any t-power-type idempotent* (*super*)*operators* (111) *has the form*

$$\mathsf{K}'(t) = \mathsf{A}_K \cdot \mathsf{K}(t) + \mathsf{U}_n(t), \tag{119}$$

where

$$\mathsf{U}_n(t) = \begin{cases} 0 & n=1, \\ \sum_{m=2}^{n} m\mathsf{K}_m t^{m-1} & n \geqslant 2. \end{cases} \tag{120}$$

Proof. To find the difference $\mathsf{U}_n(t)$ we use the expansion (111) and the band conditions for components (112)–(116). □

11. Conclusion

In general one-parametric semigroups and corresponding superoperator semigroups represented by antitriangle idempotent supermatrices and their generalization for any dimensions p, q, m, n have many unusual and nontrivial properties [18–21]. Here we considered only some of them related to their connection with functional and differential equations. It would be interesting to generalize the above constructions to higher dimensions and to study continuity properties of the introduced idempotent superoperators. These questions will be investigated elsewhere.

Acknowledgements

The author is grateful to Jan Okniński for valuable remarks and kind hospitality at the Institute of Mathematics, Warsaw University, where this work was begun. Also fruitful discussions with W. Dudek, A. Kelarev, G. Kourinnoy, W. Marcinek and B.V. Novikov are greatly acknowledged.

References

[1] E. Hille, R.S. Phillips, Functional Analysis and Semigroups, AMS, Providence, RI, 1957.
[2] E.B. Davies, One-Parameter Semigroups, Academic Press, London, 1980.
[3] J.A. Goldstein, Semigroups of Linear Operators and Applications, Oxford University Press, Oxford, 1985.
[4] E. Hille, Methods in Classical and Functional Analysis, Addison-Wesley, Reading, 1972.
[5] K.-J. Engel, R. Nagel, One-parameter Semigroups for Linear Evolution Processes, Springer-Verlag, Berlin, 1999.
[6] A. Belleni-Morante, Applied Semigroups and Evolution Equations, Oxford University Press, Oxford, 1979.
[7] D. Daners, P. Koch Medina, Abstract Evolution Equations Periodic Problems and Applications, Longman, New York, 1992.
[8] C. Berg, J.P.R. Christensen, P. Ressel, Harmonic Analysis on Semigroups, Springer-Verlag, Berlin, 1984.
[9] M. Satyanarayana, Positively Ordered Semigroups, Marcel Dekker, New York, 1979.
[10] J. Berglund, H.D. Junghenn, P. Milnes, Analysis on Semigroups, Wiley, New York, 1989.
[11] N.U. Ahmed, Semigroup Theory With Application to Systems and Control, Wiley, New York, 1991.
[12] J. Bagger, J. Wess, Partial breaking of extended supersymmetry, Phys. Lett. B 138 (1984) 105–110.
[13] P. van Nieuwenhuizen, P. West, Principles of Supersymmetry and Supergravity, Cambridge University Press, Cambridge, 1989.
[14] P. Deligne, P. Etingof, D.S. Freed, L.C. Jeffrey, D. Kazhdan, J.W. Morgan, D.R. Morrison, E. Witten (Eds.), Quantum Fields and Strings: A Cource for Mathematicians, vol. 1 & 2, AMS, Providence, RI, 1999.
[15] F.A. Berezin, Introduction to Superanalysis, Reidel, Dordrecht, 1987.
[16] D.A. Leites, Introduction to the theory of supermanifolds, Russian Math. Surv. 35 (1980) 1–64.
[17] D. Leites, Supermanifold theory, Math. Methods Sci. Invest. Petrozavodsk (1983).
[18] S. Duplij, On an alternative supermatrix reduction, Lett. Math. Phys. 37 (1996) 385–396.

[19] S. Duplij, Supermatrix representations of semigroup bands, Pure Math. Appl. 7 (1996) 235–261.
[20] S. Duplij, Semigroup methods in supersymmetric theories of elementary particles, Habilitation Thesis, Kharkov State University, math-ph/9910045, Kharkov, 1999.
[21] S. Duplij, Semisupermanifolds and semigroups, Krok, Kharkov, 2000.
[22] J. Okniński, Semigroups of Matrices, World Sci, Singapore, 1998.
[23] J.S. Ponizovskii, On a type of matrix semigroups, Semigroup Forum 44 (1992) 125–128.
[24] J. Okniński, J.S. Ponizovskii, A new matrix representation theorem for semigroups, Semigroup Forum 52 (1996) 293–305.
[25] D.V. Volkov, V.P. Akulov, Is the neutrino a Goldstone particle? Phys. Lett. B 46 (1973) 109–112.
[26] J. Wess, B. Zumino, Superspace formulation of supergravity, Phys. Lett. B 66 (1977) 361–365.
[27] A. Salam, J. Strathdee, Supersymmetry and superfields, Fortschr. Phys. 26 (1978) 57–123.
[28] S.J. Gates, M.T. Grisaru, M. Rocek, W. Ziegel, Superspace or One Thousand and One Lessons in Supersymmetry, Benjamin, Reading, 1983.
[29] B.S. DeWitt, Supermanifolds, second ed., Cambridge University Press, Cambridge, 1992.
[30] A.Y. Khrennikov, Superanalysis, Nauka, Moscow, 1997.
[31] V.S. Vladimirov, I.V. Volovich, Superanalysis. 1. Differential calculus, Theor. Math. Phys. 59 (1984) 3–27.
[32] S. Duplij, On semigroup nature of superconformal symmetry, J. Math. Phys. 32 (1991) 2959–2965.
[33] S. Duplij, Ideal structure of superconformal semigroups, Theor. Math. Phys. 106 (1996) 355–374.
[34] S. Duplij, Some abstract properties of semigroups appearing in superconformal theories, Semigroup Forum 54 (1997) 253–260.
[35] S. Duplij, On semi-supermanifolds, Pure Math. Appl. 9 (1998) 283–310.
[36] S. Duplij, W. Marcinek, Noninvertibility, semisupermanifolds and categories regularization, in: S. Duplij, J. Wess (Eds.), Noncommutative Structures in Mathematics and Physics, Kluwer, Dordrecht, 2001, pp. 125–140.
[37] S. Duplij, W. Marcinek, Semisupermanifolds and regularization of categories, modules, algebras and Yang-Baxter equation, Supersymmetry and Quantum Field Theory, Elsevier Science Publishers, Amsterdam, 2001, pp. 110–115.
[38] P. Clemént, H.J.A.M. Heijmans, S. Angenent, C.J. van Duijn, B. de Pagter, One-Parameter Semigroups, North-Holland, Amsterdam, 1987.
[39] V.G. Kac, Lie superalgebras, Adv. Math. 26 (1977) 8–96.
[40] A. Rogers, A global theory of supermanifolds, J. Math. Phys. 21 (1980) 1352–1365.
[41] D. Leites, X. Peiqi, Supersymmetry of the Schrödinger and Korteweg-de Vries operators, Univ. Stockholm preprint, hep-th/9710045, Stockholm, 1997, p. 15.
[42] N.B. Backhouse, A.G. Fellouris, Grassmann analogs of classical matrix groups, J. Math. Phys. 26 (1985) 1146–1151.
[43] L.F. Urrutia, N. Morales, The Cayley-Hamilton theorem for supermatrices, J. Phys. A 27 (1994) 1981–1997.
[44] M.Z. Nashed, Generalized Inverses and Applications, Academic Press, New York, 1976.
[45] D.L. Davis, D.W. Robinson, Generalized inverses of morphisms, Linear Algebra Appl. 5 (1972) 329–338.
[46] F. Delduc, F. Gieres, S. Gourmelen, S. Theisen, Non-standard matrix formats of Lie superalgebras, Int. J. Mod. Phys. A 14 (1999) 4043–4051.
[47] F.A. Berezin, Vvedenie v algebru i analiz s antikommutirujushimiperemennymi, Izdatelstvo MGU, Moskva, 1983.
[48] F.A. Berezin, D.A. Leites, Supermanifolds, Soviet Math. Dokl. 16 (1975) 1218–1222.
[49] R. Arnowitt, P. Nath, B. Zumino, Superfield densities and action principle in superspace, Phys. Lett. B 56 (1975) 81–86.
[50] V.F. Pakhomov, Automorphisms of tensor product of Abelian and Grassmann algebras, Mat. Zametki 16 (1974) 65–74.

[51] N.B. Backhouse, A.G. Fellouris, On the superdeterminant function for supermatrices, J. Phys. 17 (1984) 1389–1395.
[52] S.A. Duplij, O.I. Kotulska, Supermatrix structures and generalized inverses, J. Kharkov National University ser. Nuclei Particles and Fields 548 (2002) 3–14.
[53] V.N. Joshi, A determinant for rectangular matrices, Bull. Austral. Math. Soc. 21 (1980) 137–146.
[54] J.B. Hawkins, A. Ben-Israel, On generalized matrix functions, Linear and Multilinear Algebra 1 (1973) 163–171.
[55] I.J. Good, Generalized determinants and generalized variance, J. Statist. Comput. Simulation 12 (1980/81) 311–315.
[56] C.R. Rao, S.K. Mitra, Generalized Inverse of Matrices and its Application, Wiley, New York, 1971.
[57] S. Duplij, Noninvertible $N = 1$ superanalog of complex structure, J. Math. Phys. 38 (1997) 1035–1040.
[58] D. Allen, W. Windham, Operator semigroups with applications to semirings, Publ. Math. Debrecen 20 (1973) 161–175.
[59] H. Radjavi, P. Rosenthal, Simultaneous Triangularization, Springer-Verlag, Berlin, 1999.